数理解析研究所講究録
第 874 巻 1994 年 137-155

White Noise Calculus with Finite Degree of Freedom

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Introduction

It has been often said that white noise calculus is founded on an infinite dimensional analogue of Schwartz type distribution theory on a finite dimensional space. In fact, the Gelfand triple

$$(E) \subset (L^2) = L^2(E^*, \mu) \subset (E)^*$$

of white noise functionals is similar to

$$\mathcal{S}(\mathbb{R}^D) \subset L^2(\mathbb{R}^D, dx) \subset \mathcal{S}'(\mathbb{R}^D)$$

by their construction. Moreover, the formal correspondence between white noise and finite dimensional calculi (e.g., [11]) have helped us to introduce new concepts into white noise calculus successfully; for example, Fourier transform [10], infinite dimensional Laplacians [11], infinitesimal generators of infinite dimensional rotations [5], rotation-invariant operators [15], first order differential operators [18], see also [20].

The construction of white noise functionals which we have adopted as the framework of white noise calculus is due to Kubo and Takenaka [8]. The essence of their discussion is now abstracted under the name of *standard setup of white noise calculus* [5]. The axioms we use (see §2) are arranged for the operator theory on Fock space as well as for analysis of generalized white noise functionals [19], [20]. The standard setup is recapitulated in §§2-3.

Although a simple trick it is noteworthy that the "time" parameter space T can be a discrete space or even a finite set under the standard setup. If we take a finite set $T = \{1, 2, \cdots, D\}$, the corresponding white noise calculus, which is justifiably called *white noise calculus with finite degree of freedom*, yields a finite dimensional calculus based on a particular Gelfand triple

$$\mathcal{D} \subset L^2(\mathbb{R}^D, dx) \subset \mathcal{D}^*.$$

The main purpose of this paper is to study the above Gelfand triple and the resultant operator theory. In §4 we obtain a characterization of $\mathcal{D}$ and prove that $\mathcal{D}$ is a proper subspace of $\mathcal{S}(\mathbb{R}^D)$. In §5 we discuss some important operators, such as differential operators, multiplication by coordinate functions, Laplacians, infinitesimal generators of rotations and Fourier transform, by means of our operator theory on Fock space.

The present discussion would be known to some extent. In fact, Takenaka [21] attempted to explain white noise calculus by observing its one-dimensional version, namely the case of $D = 1$ in our terminology. In his quite recent work Kubo [6] discusses a discrete version of usual white noise calculus and obtains characterization of $\mathcal{D}$ in a different way. Seemingly, his original purpose is to establish an approximation theory for white noise functionals, see also [7]. What we should like to emphasize in this paper is that the fundamental features of white noise calculus do not depend on a special choice of T and E^* such as $T = \mathbf{R}$ and $E^* = \mathcal{S}'(\mathbf{R})$, but are consequences of the axioms of the standard setup.

It seems possible to generalize our discussion further in an algebraic language to make the essential structure clearer. In this connection reference to Malliavin [14], an axiomatization of Gaussian space in line with the classical work of Segal, would help us.

1 Preliminaries

We start with general notation. For a real vector space $\mathfrak{X}$ we denote its complexification by $\mathfrak{X}_{\mathbf{C}}$. Unless otherwise stated the dual space $\mathfrak{X}^*$ of a locally convex space $\mathfrak{X}$ is assumed to carry the strong dual topology. The canonical bilinear form on $\mathfrak{X}^* \times \mathfrak{X}$ is denoted by $\langle \cdot, \cdot \rangle$ or by similar symbols. When $\mathfrak{H}$ is a complex Hilbert space, in order to avoid notational confusion we do not use the hermitian inner product but the $\mathbf{C}$-bilinear form on $\mathfrak{H} \times \mathfrak{H}$.

For two locally convex spaces $\mathfrak{X}$ and $\mathfrak{Y}$ let $\mathfrak{X} \otimes_\pi \mathfrak{Y}$ denote the completetion of the algebraic tensor product $\mathfrak{X} \otimes_{\mathrm{alg}} \mathfrak{Y}$ with respect to the π-topology, i.e., the strongest locally convex topology such that the canonical bilinear map $\mathfrak{X} \times \mathfrak{Y} \to \mathfrak{X} \otimes_{\mathrm{alg}} \mathfrak{Y}$ is continuous. For two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ their Hilbert space tensor product is denoted by $\mathfrak{H} \otimes \mathfrak{K}$. It is noted that $\mathfrak{H} \otimes_\pi \mathfrak{K}$ is not isomorphic (as topological vector spaces) to the Hilbert space tensor product if they are both infinite dimensional. Nevertheless, when there is no danger of confusion, $\mathfrak{X} \otimes_\pi \mathfrak{Y}$ is also denoted by $\mathfrak{X} \otimes \mathfrak{Y}$ for simplicity. For $n \geq 1$ let $\mathfrak{X}^{\widehat{\otimes} n} \subset \mathfrak{X}^{\otimes n} = \mathfrak{X} \otimes \cdots \otimes \mathfrak{X}$ (n-times) be the closed subspace spanned by the symmetric tensors. Let $(\mathfrak{X}^{\otimes n})^*_{\mathrm{sym}}$ be the space of symmetric continuous linear functionals on $\mathfrak{X}^{\otimes n}$.

Following [19], [20] we introduce a standard countably Hilbert space just for notational convention. Let $\mathfrak{H}$ be a (real or complex) Hilbert space with norm $|\cdot|_0$ and let A be a positive selfadjoint operator on $\mathfrak{H}$ with $\inf \mathrm{Spec}\,(A) > 0$, namely, with the property that A admits a dense range and bounded inverse. Then a selfadjoint operator A^p is defined for any $p \in \mathbf{R}$ with maximal domain in $\mathfrak{H}$. Note that $\mathrm{Dom}\,(A^{-p}) = \mathfrak{H}$ for $p > 0$. We put

$$|\xi|_p = |A^p \xi|_0, \qquad \xi \in \mathrm{Dom}\,(A^p), \qquad p \in \mathbf{R}.$$

For $p \geq 0$ the vector space $\mathrm{Dom}\,(A^p)$ with the norm $|\cdot|_p$ becomes a Hilbert space which we denote by $\mathfrak{E}_p$. While, let $\mathfrak{E}_{-p}$ be the completion of $\mathfrak{H}$ with respect to $|\cdot|_{-p}$. Then these Hilbert spaces satisfy the natural inclusion relations:

$$\cdots \subset \mathfrak{E}_q \subset \cdots \subset \mathfrak{E}_p \subset \cdots \subset \mathfrak{E}_0 = \mathfrak{H} \subset \cdots \subset \mathfrak{E}_{-p} \subset \cdots \subset \mathfrak{E}_{-q} \subset \cdots, \qquad 0 \leq p \leq q.$$

Then,

$$\mathfrak{E} = \operatorname*{proj\,lim}_{p\to\infty} \mathfrak{E}_p = \bigcap_{p \geq 0} \mathfrak{E}_p$$

becomes a countably Hilbert space (abbr. CH-space) with norms $|\cdot|_p$, $p \in \mathbf{R}$. Since a general CH-space (see [1] for definition) is not necessarily of this type, we say that $\mathfrak{E}$ is the *standard* CH-space constructed from a pair $(\mathfrak{H}, \mathfrak{A})$.

It is known (see e.g., [1]) that $\mathfrak{E}^*$ (equipped with the strong dual topology) is isomorphic to the inductive limit:

$$\mathfrak{E}^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathfrak{E}_{-p} = \bigcup_{p\geq 0} \mathfrak{E}_{-p}.$$

A standard CH-space $\mathfrak{E}$ constructed from $(\mathfrak{H}, A)$ is nuclear if and only if A^{-r} is of Hilbert-Schmidt type for some $r > 0$. In that case we obtain a Gelfand triple $\mathfrak{E} \subset \mathfrak{H} \subset \mathfrak{E}^*$.

2 Standard setup – Gaussian space

Let T be a topological space with a Borel measure $\nu(dt) = dt$ and let $H = L^2(T, \nu; \mathbb{R})$ be the real Hilbert space of all ν-square integrable functions on T. The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|_0$.

Let A be a positive selfadjoint operator on H with Hilbert-Schmidt inverse. Then there exist an increasing sequence of positive numbers $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and a complete orthonormal basis $(e_j)_{j=0}^{\infty}$ for H such that $Ae_j = \lambda_j e_j$ and

$$\delta \equiv \left(\sum_{j=0}^{\infty} \lambda_j^{-2} \right)^{1/2} = \| A^{-1} \|_{\mathrm{HS}} < \infty.$$

Let E be the standard CH-space constructed from (H, A). By definition the norms are given by

$$|\xi|_p = |A^p \xi|_0 = \left(\sum_{j=0}^{\infty} \lambda_j^{2p} \langle \xi, e_j \rangle^2 \right)^{1/2}, \qquad \xi \in E, \quad p \in \mathbb{R}.$$

Since A^{-1} is of Hilbert-Schmidt type by assumption, E becomes a nuclear Fréchet space and we obtain a Gelfand triple

$$E \subset H = L^2(T, \nu; \mathbb{R}) \subset E^*.$$

The canonical bilinear form on $E^* \times E$ is also denoted by $\langle \cdot, \cdot \rangle$.

By construction each $\xi \in E$ is a function on T determined up to ν-null functions. This hinders us from introducing a delta-function which is indispensable to our discussion. Accordingly we are led to the following:

(H1) For each $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T such that $\xi(t) = \tilde{\xi}(t)$ for ν-a.e. $t \in T$.

Once this is satisfied, we always assume that every element in E is a continuous function on T and do not use the symbol $\tilde{\xi}$. We further need:

(H2) For each $t \in T$ a linear functional $\delta_t : \xi \mapsto \xi(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$;

(H3) The map $t \mapsto \delta_t \in E^*$, $t \in T$, is continuous.

(Recall that E^* carries the strong dual topology.) Under (H1)-(H2) the convergence in E implies the pointwise convergence as functions on T. If we have (H3) in addition, the convergence is uniform on every compact subset of T. Moreover, it is noted that the properties (H1)-(H3) are preserved under forming tensor products, see [17].

For another reason (see §3) we need one more assumption:

(S) $\lambda_0 = \inf \operatorname{Spec}(A) > 1$.

The constant number

$$0 < \rho \equiv \lambda_0^{-1} = \| A^{-1} \|_{\mathrm{OP}} < 1$$

is important as well as δ in deriving various inequalities, though we do not use them explicitly in this paper.

By the Bochner-Minlos theorem there exists a unique probability measure μ on E^* (equipped with the Borel σ-field) such that

$$\exp\left(-\frac{1}{2} | \xi |_0^2\right) = \int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \qquad \xi \in E.$$

This μ is called the *Gaussian measure* and the probability space (E^*, μ) is called the *Gaussian space.*

3 Standard setup – White noise functionals

We shall construct test and generalized functions on the Gaussian space (E^*, μ) by means of standard CH-spaces. As usual we put $(L^2) = L^2(E^*, \mu; \mathbb{C})$ for simplicity.

The canonical bilinear form on $(E^{\otimes n})^* \times (E^{\otimes n})$ is denoted by $\langle \cdot, \cdot \rangle$ again and its $\mathbb{C}$-bilinear extension to $(E_{\mathbb{C}}^{\otimes n})^* \times (E_{\mathbb{C}}^{\otimes n})$ is also denoted by the same symbol. For $x \in E^*$ let $: x^{\otimes n} : \in (E^{\otimes n})^*_{\mathrm{sym}}$ be defined uniquely as

$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \qquad \xi \in E_{\mathbb{C}}. \tag{1}$$

The explicit form of $: x^{\otimes n} :$ is well known, see e.g., [4], [17], [20]. The function ϕ_ξ will be referred to as an *exponential vector.*

By virtue of the celebrated Wiener-Itô decomposition theorem, with each $\phi \in (L^2)$ we may associate a unique sequence $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$, $n = 0, 1, 2, \cdots$, such that

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, f_n \right\rangle, \qquad x \in E^*, \tag{2}$$

where the bilinear forms and the convergence of the series are understood in the L^2-sense. Moreover, (2) is an orthogonal direct sum and

$$\| \phi \|_0^2 \equiv \int_{E^*} |\phi(x)|^2 \mu(dx) = \sum_{n=0}^{\infty} n! \, | f_n |_0^2 .$$

In other words, we have established a unitary isomorphism between (L^2) and the Boson Fock space over $H_{\mathbb{C}}$.

We then need a second quantized operator $\Gamma(A)$, where A is the same operator as we used to construct $E \subset H = L^2(T, \nu; \mathbb{R}) \subset E^*$. For $\phi \in (L^2)$ given as in (2) we put

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, A^{\otimes n} f_n \right\rangle .$$

Equipped with the maximal domain, $\Gamma(A)$ becomes a positive selfadjoint operator on (L^2), and thereby the pair $((L^2), \Gamma(A))$ yields a standard CH-space which we shall denote by

(E). Since $\Gamma(A)$ admits Hilbert-Schmidt inverse by the hypothesis (S), the space (E) is a nuclear Fréchet space and

$$(E) \subset (L^2) = L^2(E^*, \mu; \mathbb{C}) \subset (E)^*$$

becomes a complex Gelfand triple. Elements in (E) and $(E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$ and by $\| \cdot \|_p$ the norm introduced from $\Gamma(A)$, namely,

$$\| \phi \|_p^2 = \| \Gamma(A)^p \phi \|_0^2 = \sum_{n=0}^{\infty} n! \left| (A^{\otimes n})^p f_n \right|_0^2 = \sum_{n=0}^{\infty} n! \, | f_n |_p^2, \qquad \phi \in (E),$$

where ϕ and $(f_n)_{n=0}^{\infty}$ are related as in (2).

By construction each $\phi \in (E)$ is defined only up to μ-null functions. However, it follows from Kubo-Yokoi's continuous version theorem [9] (see also [17]) that for $\phi \in (E)$ the right hand side of (2) converges absolutely at each $x \in E^*$ and becomes a unique continuous function on E^* which coincides with $\phi(x)$ for μ-a.e. $x \in E^*$. Thus, (E) is always assumed to be a space of continuous functions on E^* and for $\phi \in (E)$ the right hand side of (2) is understood as pointwisely convergent series as well as in the sense of norms $\| \cdot \|_p$.

It is known that $\phi_\xi \in (E)$ for any $\xi \in E_{\mathbb{C}}$. The S-transform of $\Phi \in (E)^*$ is a function on $E_{\mathbb{C}}$ defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle = e^{-\langle \xi, \xi \rangle/2} \int_{E^*} \Phi(x) e^{\langle x, \xi \rangle} \mu(dx), \qquad \xi \in E_{\mathbb{C}}. \tag{3}$$

On the ther hand, the T-transform is defined by

$$T\Phi(\xi) = \left\langle\!\left\langle \Phi, e^{i\langle \cdot, \xi \rangle} \right\rangle\!\right\rangle = \int_{E^*} \Phi(x) e^{i\langle x, \xi \rangle} \mu(dx), \qquad \xi \in E_{\mathbb{C}}. \tag{4}$$

Of course the integral expressions are valid only when the integrands are integrable functions, in paticular when $\Phi \in (E)$. There is a simple relation:

$$T\Phi(\xi) = S\Phi(i\xi) e^{-\langle \xi, \xi \rangle/2}, \qquad S\Phi(\xi) = T\Phi(-i\xi) e^{-\langle \xi, \xi \rangle/2}, \qquad \xi \in E_{\mathbb{C}}. \tag{5}$$

4 Reduction to finite degree of freedom

From now on let $T = \{1, 2, \cdots, D\}$ be a finite set with discrete topology and counting measure ν. Then $H = L^2(T, \nu; \mathbb{R}) \cong \mathbb{R}^D$ under the natural identification. The L^2-norm and the Euclidean norm coincide:

$$|\xi|^2 = \sum_{j=1}^{D} |\xi_j|^2, \qquad \xi = (\xi_1, \cdots, \xi_D) \in H. \tag{6}$$

In this context the operator A needed to construct Gaussian space is merely a symmetric matrix with eigenvalues $1 < \lambda_1 \le \cdots \le \lambda_D$. The corresponding unit eigenvectors are denoted by $e_1, \cdots, e_D$. Then, by definition

$$| \xi |_p^2 = \sum_{j=1}^{D} \lambda_j^{2p} \langle \xi, e_j \rangle^2, \qquad \xi \in H = \mathbb{R}^D.$$

Since $\lambda_1 |\xi|_p \le |\xi|_{p+1} \le \lambda_D |\xi|_p$ for $\xi \in \mathbb{R}^D$, all the norms $|\cdot|_p$ are equivalent and we use only the Euclidean norm (6). Note also that $|\xi| = |\xi|_0$ for $\xi \in \mathbb{R}^D$. Moreover, the corresponding Gelfand triple becomes $E = H = E^* = \mathbb{R}^D$.

Since T is a discrete space and ν is a counting measure on it, the verification of the hypotheses (H1)-(H3) is very simple. The evaluation map $\delta_j : \xi = (\xi_1, \cdots, \xi_D) \mapsto \xi_j \in \mathbb{R}$ is merely a coordinate projection. Hence $\delta_j \in (\mathbb{R}^D)^*$ and

$$\delta_j = (0, \cdots, 0, \overset{j\text{-th}}{1}, 0, \cdots, 0), \qquad j = 1, 2, \cdots, D, \tag{7}$$

through the canonical bilinear form $\langle \cdot, \cdot \rangle$ on $(\mathbb{R}^D)^* \times \mathbb{R}^D$.

The Gaussian measure μ on $E^* = \mathbb{R}^D$ is nothing but the product of 1-dimensional standard Gaussian measures:

$$\mu(dx) = \left(\frac{1}{\sqrt{2\pi}} \right)^D e^{-|x|^2/2} dx,$$

where $dx = dx_1 \cdots dx_D$, $x = (x_1, \cdots, x_D) \in \mathbb{R}^D$. Then, by means of $\Gamma(A)$ we obtain the Gelfand triple of white noise functionals with finite degree of freedom:

$$(E) \subset (L^2) = L^2(\mathbb{R}^D, \mu; \mathbb{C}) \subset (E)^*.$$

By the continuous version theorem (E) is a space of continuous functions on $\mathbb{R}^D$. We shall study (E) in more detail.

Lemma 4.1 *Any polynomial belongs to (E).*

PROOF. In general, it follows from the definition (1) that

$$\left\langle :x^{\otimes n}:, \xi^{\otimes n} \right\rangle = \frac{|\xi|^n}{2^{n/2}} H_n \left(\frac{\langle x, \xi \rangle}{\sqrt{2}|\xi|} \right), \qquad \xi \in E, \quad \xi \neq 0,$$

where H_n is the Hermite polynomial of degree n. Putting $\xi = \delta_j$, we obtain

$$\left\langle :x^{\otimes n}:, \delta_j^{\otimes n} \right\rangle = \frac{1}{2^{n/2}} H_n \left(\frac{x_j}{\sqrt{2}} \right) = x_j^n + \cdots.$$

Hence (E) contains every polynomial in x_j and therefore in $x_1, \cdots, x_D$ since (E) is closed under pointwise multiplication. qed

Lemma 4.2 *Let F be a $\mathbb{C}$-valued function on $\mathbb{C}^D$. Then there exists some $\phi \in (E)$ such that $F = S\phi$ if and only if*

(i) *F is entire holomorphic on $\mathbb{C}^D$;*

(ii) *for any $\epsilon > 0$ there exists $C \ge 0$ such that $|F(\xi)| \le C e^{\epsilon |\xi|^2}$, $\xi \in \mathbb{C}^D$.*

In that case ϕ is unique.

This is a simple consequence of the characterization theorem for white noise test functionals [13], see also [6, Theorem 3.4]. Here is notation for simplicity. For a function ϕ on $\mathbb{R}^D$ we put

$$\phi^\alpha(x) = \phi(\alpha x), \qquad \alpha > 0, \quad x \in \mathbb{R}^D.$$

It is known that $\phi^\alpha \in (E)$ for any $\alpha > 0$ and $\phi \in (E)$.

Lemma 4.3 *If $\phi \in (E)$, then $\phi \cdot e^{-\epsilon|x|^2} \in L^1(\mathbb{R}^D, dx)$ for any $\epsilon > 0$.*

PROOF. In fact,

$$\begin{aligned}\int_{\mathbb{R}^D} |\phi(x)| e^{-\epsilon|x|^2} dx &= \left(\frac{1}{\sqrt{2\epsilon}}\right)^D \int_{\mathbb{R}^D} \left|\phi\left(\frac{x}{\sqrt{2\epsilon}}\right)\right| e^{-|x|^2/2} dx \\ &= \left(\frac{\sqrt{2\pi}}{\sqrt{2\epsilon}}\right)^D \int_{\mathbb{R}^D} \left|\phi^{1/\sqrt{2\epsilon}}(x)\right| \mu(dx) < \infty,\end{aligned}$$

since $\phi^{1/\sqrt{2\epsilon}} \in (E) \subset L^2(\mathbb{R}^D, \mu) \subset L^1(\mathbb{R}^D, \mu)$. qed

Lemma 4.4 *Let ϕ be a $\mathbb{C}$-valued function on $\mathbb{R}^D$. If $\phi \cdot e^{-\epsilon|x|^2} \in L^1(\mathbb{R}^D, dx)$ for any $\epsilon > 0$, the Fourier transform*

$$(\phi \cdot e^{-\epsilon|x|^2})\widehat{\ }(\xi) = \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \phi(x) e^{-\epsilon|x|^2} e^{i\langle x, \xi\rangle} dx,$$

converges absolutely at any $\xi \in \mathbb{C}^D$ and becomes an entire holomorphic function on $\mathbb{C}^D$.

PROOF. We first prove that the integral

$$\int_{\mathbb{R}^D} \phi(x) e^{-\epsilon|x|^2} e^{i\langle x, \xi\rangle} dx \tag{8}$$

converges absolutely at any $\xi \in \mathbb{C}^D$. Suppose $\xi = \xi_1 + i\xi_2$ with $\xi_1, \xi_2 \in \mathbb{R}^D$ and take $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 + \epsilon_2 = \epsilon$. In view of the obvious inequality:

$$e^{-\epsilon_2|x|^2 - \langle x, \xi_2\rangle} = \exp\left(-\epsilon_2 \left| x + \frac{\xi_2}{2\epsilon_2}\right|^2 + \frac{|\xi_2|^2}{4\epsilon_2}\right) \le e^{|\xi_2|^2/4\epsilon_2}, \qquad x \in \mathbb{R}^D,$$

we see that

$$\begin{aligned}\int_{\mathbb{R}^D} \left|\phi(x) e^{-\epsilon|x|^2} e^{i\langle x, \xi\rangle}\right| dx &= \int_{\mathbb{R}^D} \left|\phi(x) e^{-\epsilon|x|^2} e^{i\langle x, \xi_1\rangle} e^{-\langle x, \xi_2\rangle}\right| dx \\ &= \int_{\mathbb{R}^D} |\phi(x)| e^{-\epsilon_1|x|^2} e^{-\epsilon_2|x|^2 - \langle x, \xi_2\rangle} dx \\ &\le e^{|\xi_2|^2/4\epsilon_2} \int_{\mathbb{R}^D} |\phi(x)| e^{-\epsilon_1|x|^2} dx < \infty\end{aligned}$$

by asssumption. Hence (8) converges absolutely at any $\xi \in \mathbb{C}^D$.

For holomorphy it is sufficient to show that

$$\int_{\mathbb{R}^D} \phi(x) e^{-\epsilon|x|^2} i x_j e^{i\langle x, \xi\rangle} dx, \qquad j = 1, 2, \cdots, D,$$

converges absolutely and uniformly on every compact neighborhood of $\xi \in \mathbb{C}^D$. We put $\xi = \xi_1 + i\xi_2$, $\xi_1, \xi_2 \in \mathbb{R}^D$ and take $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ with $\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3$. A similar argument as above leads us to the following

$$\begin{aligned}\int_{\mathbb{R}^D} \left|\phi(x) e^{-\epsilon|x|^2} i x_j e^{i\langle x, \xi\rangle}\right| dx &= \int_{\mathbb{R}^D} |\phi(x)| e^{-\epsilon_1|x|^2} |x_j| e^{-\epsilon_2|x|^2} e^{-\epsilon_3|x|^2 - \langle x, \xi_2\rangle} dx \\ &\le e^{|\xi_2|^2/4\epsilon_3} \max_{x \in \mathbb{R}^D} |x_j| e^{-\epsilon_2|x|^2} \int_{\mathbb{R}^D} |\phi(x)| e^{-\epsilon_1|x|^2} dx.\end{aligned}$$

Then the desired assertion is straightforward. qed

Proposition 4.5 *A continuous function $\phi : \mathbb{R}^D \to \mathbb{C}$ belongs to (E) if and only if*
(i) $\phi \cdot e^{-\epsilon|x|^2} \in L^1(\mathbb{R}^D, dx)$ *for any* $\epsilon > 0$;
(ii) *for any* $\epsilon > 0$ *there exists* $C \geq 0$ *such that*

$$\left| e^{\langle \xi, \xi \rangle /2} (\phi \cdot e^{-|x|^2/2})\hat{}(\xi) \right| \leq C e^{\epsilon |\xi|^2}, \qquad \xi \in \mathbb{C}^D.$$

PROOF. Suppose first that $\phi \in (E)$. Then (i) follows from Lemma 4.3. By definition for $\xi \in \mathbb{R}^D$,

$$(\phi \cdot e^{-|x|^2/2})\hat{}(\xi) = \left(\frac{1}{\sqrt{2\pi}} \right)^D \int_{\mathbb{R}^D} \phi(x) e^{-|x|^2/2} e^{i\langle x, \xi \rangle} dx = \int_{\mathbb{R}^D} \phi(x) e^{i\langle x, \xi \rangle} \mu(dx).$$

Hence, in view of Lemma 4.4 and the definition of T-transform (4) we obtain

$$(\phi \cdot e^{-|x|^2/2})\hat{}(\xi) = T\phi(\xi), \qquad \xi \in \mathbb{C}^D.$$

Therefore, by (5) we see that

$$S\phi(i\xi) = T\phi(\xi) e^{\langle \xi, \xi \rangle /2} = e^{\langle \xi, \xi \rangle /2} (\phi \cdot e^{-|x|^2/2})\hat{}(\xi), \qquad \xi \in \mathbb{C}^D.$$

Then, it is easily seen that (ii) is merely reformulation of the boundedness condition of $S\phi$ in Lemma 4.2 (ii).

Conversely, (i) implies the holomorphy of $(\phi \cdot e^{-|x|^2/2})\hat{}$ by Lemma 4.4, and therefore of $e^{-\langle \xi, \xi \rangle /2} (\phi \cdot e^{-|x|^2/2})\hat{}(-i\xi)$. Then (ii) guarantees the existence of $\psi \in (E)$ such that

$$S\psi(\xi) = e^{-\langle \xi, \xi \rangle /2} (\phi \cdot e^{-|x|^2/2})\hat{}(-i\xi), \qquad \xi \in \mathbb{C}^D, \tag{9}$$

by Lemma 4.2. On the other hand,

$$\begin{aligned} S\psi(\xi) &= T\psi(-i\xi) e^{-\langle \xi, \xi \rangle /2} \\ &= e^{-\langle \xi, \xi \rangle /2} \int_{\mathbb{R}^D} \psi(x) e^{i\langle x, -i\xi \rangle} \mu(dx) \\ &= e^{-\langle \xi, \xi \rangle /2} (\psi \cdot e^{-|x|^2/2})\hat{}(-i\xi), \qquad \xi \in \mathbb{C}^D. \end{aligned} \tag{10}$$

In view of (9) and (10) we obtain

$$(\phi \cdot e^{-|x|^2/2})\hat{}(\xi) = (\psi \cdot e^{-|x|^2/2})\hat{}(\xi), \qquad \xi \in \mathbb{R}^D. \tag{11}$$

Note that $\phi \cdot e^{-|x|^2/2}$ belongs to $L^1(\mathbb{R}^D, dx)$ by assumption and so does $\psi \cdot e^{-|x|^2/2}$ by Lemma 4.3. Since the Fourier transform of an L^1-function is unique, it follows from (11) that $\phi = \psi$ and hence $\phi \in (E)$. qed

There is a natural unitary isomorphism from $L^2(\mathbb{R}^D, \mu)$ onto $L^2(\mathbb{R}^D, dx)$ given by

$$U\phi(x) = \left(\frac{1}{\sqrt{2\pi}} \right)^{D/2} e^{-|x|^2/4} \phi(x), \qquad \phi \in L^2(\mathbb{R}^D, \mu). \tag{12}$$

Let $\mathcal{D}$ denote the image of (E) under the unitary map U. Then, the Gelfand triple $(E) \subset L^2(\mathbb{R}^D, \mu) \subset (E)^*$ yields a new Gelfand triple

$$\mathcal{D} \subset L^2(\mathbb{R}^D, dx) \subset \mathcal{D}^*.$$

This is the basis of finite dimensional calculus derived from white noise calculus with finite degree of freedom. As an immediate consequence of Proposition 4.5 we have

Theorem 4.6 *A continuous function* $\psi : \mathbb{R}^D \to \mathbb{C}$ *belongs to* $\mathcal{D}$ *if and only if*
(i) $\psi \cdot e^{\left(\frac{1}{4}-\epsilon\right)|x|^2} \in L^1(\mathbb{R}^D, dx)$ *for any* $\epsilon > 0$;
(ii) *for any* $\epsilon > 0$ *there exists* $C \geq 0$ *such that*

$$\left| e^{\langle \xi, \xi \rangle /2} (\psi \cdot e^{-|x|^2/4})\widehat{\ }(\xi) \right| \leq C e^{\epsilon |\xi|^2}, \qquad \xi \in \mathbb{C}^D.$$

We next prove the following

Proposition 4.7 *If* $\phi \in (E)$, *then* $\phi \cdot e^{-\epsilon|x|^2} \in \mathcal{S}(\mathbb{R}^D)$ *for any* $\epsilon > 0$.

PROOF. Since $\mathcal{S}(\mathbb{R}^D)$ is invariant under the Fourier transform, it is sufficient to prove that $(\phi \cdot e^{-\epsilon|x|^2})\widehat{\ } \in \mathcal{S}(\mathbb{R}^D)$. It follows from Lemmas 4.3 and 4.4 that $(\phi \cdot e^{-\epsilon|x|^2})\widehat{\ }$ is an entire holomorphic function on $\mathbb{C}^D$ and therefore belongs to $C^\infty(\mathbb{R}^D)$. For a polynomial $P(x) = P(x_1, \cdots, x_D)$ we write

$$P(\partial) = P\left(\frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_D}\right)$$

for simplicity. Then, modelled after the proof of Lemma 4.4, one can easily see that

$$\left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \phi(x) e^{-\epsilon|x|^2} P(\partial) e^{i\langle x, \xi \rangle} dx = \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \phi(x) e^{-\epsilon|x|^2} P(ix) e^{i\langle x, \xi \rangle} dx$$

converges absolutely and uniformly on every compact neighborhood of $\xi \in \mathbb{C}^D$. Hence

$$P(\partial)(\phi \cdot e^{-\epsilon|x|^2})\widehat{\ }(\xi) = (\phi P(ix) e^{-\epsilon|x|^2})\widehat{\ }(\xi), \qquad \xi \in \mathbb{C}^D. \tag{13}$$

On the other hand, since $P(ix)$ belongs to (E) by Lemma 4.1 and (E) is closed under multiplication, $\phi_1(x) = \phi(x)P(ix)$ belongs to (E) as well. Then (13) becomes

$$\begin{aligned} P(\partial)(\phi \cdot e^{-\epsilon|x|^2})\widehat{\ }(\xi) &= (\phi_1 \cdot e^{-\epsilon|x|^2})\widehat{\ }(\xi) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \phi_1(x) e^{-\epsilon|x|^2} e^{i\langle x, \xi \rangle} dx \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^D \left(\frac{1}{\sqrt{2\epsilon}}\right)^D \int_{\mathbb{R}^D} \phi_1\left(\frac{x}{\sqrt{2\epsilon}}\right) e^{-|x|^2/2} e^{i\langle x/\sqrt{2\epsilon}, \xi \rangle} dx \\ &= \left(\frac{1}{\sqrt{2\epsilon}}\right)^D \int_{\mathbb{R}^D} \phi_1^{\sqrt{2\epsilon}}(x) e^{i\langle x, \xi/\sqrt{2\epsilon} \rangle} \mu(dx). \end{aligned}$$

Hence by (4) and (5) we have

$$\begin{aligned} P(\partial)(\phi \cdot e^{-\epsilon|x|^2})\widehat{\ }(\xi) &= \left(\frac{1}{\sqrt{2\epsilon}}\right)^D T\phi_1^{1/\sqrt{2\epsilon}}\left(\frac{\xi}{\sqrt{2\epsilon}}\right) \\ &= \left(\frac{1}{\sqrt{2\epsilon}}\right)^D S\phi_1^{1/\sqrt{2\epsilon}}\left(\frac{i\xi}{\sqrt{2\epsilon}}\right) e^{-\langle \xi/\sqrt{2\epsilon}, \xi/\sqrt{2\epsilon} \rangle /2}. \end{aligned} \tag{14}$$

Since $\phi_1^{1/\sqrt{2\epsilon}} \in (E)$, it follows from Lemma 4.2 that there exists $C \geq 0$ such that

$$\left| S\phi_1^{1/\sqrt{2\epsilon}}(\xi) \right| \leq C e^{|\xi|^2/4}, \qquad \xi \in \mathbb{C}^D.$$

Hence we have

$$\left|S\phi_1^{1/\sqrt{2\epsilon}}\left(\frac{i\xi}{\sqrt{2\epsilon}}\right)\right| \le Ce^{|\xi|^2/8\epsilon}, \qquad \xi \in \mathbb{C}^D.$$

Then, in view of (14) we see that for $\xi \in \mathbb{R}^D$,

$$\left|P(\partial)(\phi \cdot e^{-\epsilon|x|^2}\widehat{)}(\xi)\right| \le \left(\frac{1}{\sqrt{2\epsilon}}\right)^D Ce^{|\xi|^2/8\epsilon}e^{-|\xi|^2/4\epsilon} = \frac{C}{(2\epsilon)^{D/2}}e^{-|\xi|^2/8\epsilon}, \qquad \xi \in \mathbb{R}^D.$$

Then for another polynomial Q it holds that

$$\left|Q(\xi)P(\partial)(\phi \cdot e^{-\epsilon|x|^2}\widehat{)}(\xi)\right| \le \frac{C}{(2\epsilon)^{D/2}}|Q(\xi)|e^{-|\xi|^2/8\epsilon} \longrightarrow 0$$

as $|\xi| \to \infty$, $\xi \in \mathbb{R}^D$. Consequently, $(\phi \cdot e^{-\epsilon|x|^2}\widehat{)} \in \mathcal{S}(\mathbb{R}^D)$. qed

Corollary 4.8 *$\mathcal{D} \subset \mathcal{S}(\mathbb{R}^D)$ and $\mathcal{D} \ne \mathcal{S}(\mathbb{R}^D)$.*

PROOF. The inclusion is immediate from Proposition 4.7. As for $\mathcal{D} \ne \mathcal{S}(\mathbb{R}^D)$ we need only to apply Theorem 4.6 to $\psi(x) = e^{-|x|^2/8}$. qed

The above result is obtained also by Kubo [6, Theorem 3.5].

5 Corresponding operators

In the theory of operators on white noise functionals a principal role is played by annihilation (Hida's differential) and creation operators. In our context Hida's differential operator is defined by

$$\partial_j\phi(x) = \lim_{\theta\to 0}\frac{\phi(x+\theta\delta_j)-\phi(x)}{\theta}, \qquad \phi \in (E), \quad x \in \mathbb{R}^D.$$

Then one sees immediately from (7) that

$$\partial_j = \frac{\partial}{\partial x_j}, \qquad j = 1, 2, \cdots, D.$$

A creation operator is its adjoint with respect to the Gaussian measure μ.

Lemma 5.1 *$\partial_j^* = x_j - \partial_j$ and $[\partial_j, \partial_k^*] = \delta_{jk}$.*

PROOF. Here is a direct proof though the assertion is entirely clear from general theory. Let $\phi, \psi \in (E)$. Then, by definition,

$$\langle\langle \partial_j^*\phi,\, \psi\rangle\rangle = \langle\langle \phi,\, \partial_j\psi\rangle\rangle = \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \partial_j\psi(x)\cdot\phi(x)e^{-|x|^2/2}dx. \tag{15}$$

By partial integration we have

$$\int_{-\infty}^{\infty} \partial_j\psi(x)\cdot\phi(x)e^{-|x|^2/2}dx_j =$$
$$= \psi(x)\phi(x)e^{-|x|^2/2}\Big|_{x_j=-\infty}^{\infty} - \int_{-\infty}^{\infty}\psi(x)\left\{\partial_j\phi(x) - \phi(x)x_j\right\}e^{-|x|^2/2}dx_j.$$

The first term vanishes since $\psi(x)\phi(x)e^{-|x|^2/2} \in \mathcal{S}(\mathbb{R}^D)$ by Proposition 4.7. Hence (15) becomes

$$\begin{aligned}\langle\langle \partial_j^* \phi, \psi \rangle\rangle &= \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \psi(x)\,(-\partial_j\phi(x) + \phi(x)x_j)\, e^{-|x|^2/2} dx \\ &= -\langle\langle \partial_j\phi, \psi \rangle\rangle + \langle\langle x_j\phi, \psi \rangle\rangle\,.\end{aligned}$$

This completes the proof. qed

It follows from the general theory that $\partial_j \in \mathcal{L}((E),(E))$ and $\partial_j^* \in \mathcal{L}((E)^*,(E)^*)$. In our case of finite degree of freedom, it is easily verified that $\partial_j^* \in \mathcal{L}((E),(E))$ as well. This is because $\delta_j \in E$ though $\delta_t \in E^*$ in a usual case.

Using the unitary operator $U : L^2(\mathbb{R}^D, \mu) \to L^2(\mathbb{R}^D, dx)$ introduced in (12), we study a few interesting operators in $\mathcal{L}((E),(E)^*)$. Note that if $\Xi \in \mathcal{L}((E),(E)^*)$ then $U\Xi U^{-1} \in \mathcal{L}(\mathcal{D},\mathcal{D}^*)$. We begin with the following

Proposition 5.2

$$U\partial_j U^{-1} = \frac{x_j}{2} + \frac{\partial}{\partial x_j}, \qquad U\partial_j^* U^{-1} = \frac{x_j}{2} - \frac{\partial}{\partial x_j}, \qquad U x_j U^{-1} = x_j.$$

In particular,

$$P_j = \frac{1}{2i}(U\partial_j U^{-1} - U\partial_j^* U^{-1}) = \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad Q_j = U\partial_j U^{-1} + U\partial_j^* U^{-1} = x_j$$

are the Schrödinger representation of CCR on $L^2(\mathbb{R}^D, dx)$ with common domain $\mathcal{D}$.

PROOF. For $\psi \in \mathcal{D}$ we have by definition

$$U\partial_j U^{-1}\psi(x) = e^{-|x|^2/4}\frac{\partial}{\partial x_j}\left(e^{|x|^2/4}\psi(x)\right) = \frac{x_j}{2}\psi(x) + \frac{\partial\psi}{\partial x_j}(x).$$

Using an obvious relation $Ux_jU^{-1} = x_j$, we come to

$$U\partial_j^* U^{-1} = U(x_j - \partial_j)U^{-1} = x_j - \left(\frac{x_j}{2} + \frac{\partial}{\partial x_j}\right) = \frac{x_j}{2} - \frac{\partial}{\partial x_j}.$$

The rest is apparent. qed

In our case of finite degree of freedom an integral kernel operator [5] is merely a *finite* linear combination of compositions of creation and annihilation operators with normal ordering:

$$\Xi_{l,m}(\kappa) = \sum \kappa(i_1,\cdots,i_l,j_1,\cdots,j_m)\partial_{i_1}^*\cdots\partial_{i_l}^*\partial_{j_1}\cdots\partial_{j_m}, \tag{16}$$

where $i_1,\cdots,i_l,j_1,\cdots,j_m$ run over $T = \{1,2,\cdots,D\}$. Using Lemma 5.1 one observes that $\Xi_{l,m}(\kappa)$ is a finite linear combination of differential operators with polynomial coefficients:

$$\Xi_{l,m}(\kappa) = \sum_{\substack{|\alpha|\le l \\ |\beta|\le l+m}} C(\alpha,\beta)x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta,$$

with multi-indices $\alpha = (\alpha_1, \cdots, \alpha_D)$, $\beta = (\beta_1, \cdots, \beta_D)$. On the other hand, it follows from Proposition 5.2 that $U\Xi_{l,m}(\kappa)U^{-1}$ is again a finite linear combination of differential operators with polynomial coefficients:

$$U\Xi_{l,m}(\kappa)U^{-1} = \sum_{\substack{|\alpha|\le l+m \\ |\beta|\le l+m}} C(\alpha,\beta)x^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}, \tag{17}$$

or in terms of the operators P_j and Q_j introduced in Proposition 5.2:

$$U\Xi_{l,m}(\kappa)U^{-1} = \sum_{\substack{|\alpha|\le l+m \\ |\beta|\le l+m}} C(\alpha,\beta)Q^{\alpha}P^{\beta}. \tag{18}$$

The theory of Fock expansion ([16], [19], [20]) says that every operator $\Xi \in \mathcal{L}((E),(E)^*)$ admits an infinite series expansion in terms of integral kernel operators:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}). \tag{19}$$

The meaning of convergence is discussed in detail, see the above quoted papers. Thus, every operator in $\mathcal{L}(\mathcal{D},\mathcal{D}^*)$ is expressed in an infinite linear combination of operators of the form (17) or equivalently (18). Inserting (18) into (19) we obtain

$$U\Xi U^{-1} = \sum_{l,m=0}^{\infty} \sum_{\substack{|\alpha|\le l+m \\ |\beta|\le l+m}} C_{l,m}(\alpha,\beta)Q^{\alpha}P^{\beta}. \tag{20}$$

Formally we may rearrange the infinite series (20) according to the usual order of multi-index notation:

$$U\Xi U^{-1} = \sum_{\alpha,\beta} C(\alpha,\beta)Q^{\alpha}P^{\beta},$$

though the meaning of the convergence becomes unclear. In this sense the Fock expansion is more complete! Incidentally we note that (20) leads us to a statement of "irreducibility" of the Schrödinger representation of CCR on $L^2(\mathbb{R}^D, dx)$, where the common domain of P_j and Q_j is taken to be $\mathcal{D}$.

The Gross Laplacian and the number operator are defined respectively by

$$\Delta_G = \sum_{j=1}^{D} \partial_j^2, \qquad N = \sum_{j=1}^{D} \partial_j^*\partial_j.$$

Since

$$x_j^2 = (\partial_j^* + \partial_j)^2 = \partial_j^{*2} + \partial_j^2 + \partial_j^*\partial_j + \partial_j\partial_j^* = \partial_j^{*2} + \partial_j^2 + 2\partial_j^*\partial_j + 1$$

by Lemma 5.1, we have

$$\sum_{j=1}^{D}(x_j^2 - 1) = \Delta_G^* + \Delta_G + 2N.$$

The left hand side is "renormalized" Euclidean norm which arises naturally in case of infinite degree of freedom, see [15].

Proposition 5.3 *It holds that*

$$\begin{aligned} U\Delta_G U^{-1} &= \sum_{j=1}^{D}\left(\frac{\partial^2}{\partial x_j^2}+x_j\frac{\partial}{\partial x_j}+\frac{x_j^2}{4}+\frac{1}{2}\right),\\ U\Delta_G^* U^{-1} &= \sum_{j=1}^{D}\left(\frac{\partial^2}{\partial x_j^2}-x_j\frac{\partial}{\partial x_j}+\frac{x_j^2}{4}-\frac{1}{2}\right),\\ UNU^{-1} &= \sum_{j=1}^{D}\left(-\frac{\partial^2}{\partial x_j^2}+\frac{x_j^2}{4}-\frac{1}{2}\right). \end{aligned}$$

The proof is straightforward from Proposition 5.2. On the other hand, for the usual Laplacian

$$\Delta=\sum_{j=1}^{D}\frac{\partial^2}{\partial x_j^2}$$

on $L^2(\mathbb{R}^D, dx)$ we have

$$U^{-1}\Delta U=\sum_{j=1}^{D}\left(\partial_j^2-x_j\partial_j+\frac{x_j^2}{4}-\frac{1}{2}\right)=\sum_{j=1}^{D}\left(-\partial_j^*\partial_j+\frac{x_j^2}{4}-\frac{1}{2}\right).$$

This expression motivated Umemura [22] to introduce an infinite dimensional Laplacian (in our terminology $-N$) by omitting the divergent terms $\frac{x_j^2}{4}-\frac{1}{2}$.

As is shown in [5], every infinitesimal generator of a regular one-parameter subgroup $\{g_\theta\}_{\theta\in\mathbb{R}}$ of $O(E;H)$ is expressed in the form:

$$\left.\frac{d}{d\theta}\right|_{\theta=0}\Gamma(g_\theta)=\int_{T\times T}\kappa(s,t)(x(s)\partial_t-x(t)\partial_s)dsdt,$$

where $\kappa\in(E\otimes E)^*$ is a skew-symmetric distribution. Thus,

$$x(s)\partial_t-x(t)\partial_s=\partial_s^*\partial_t-\partial_t^*\partial_s$$

is regarded as an infinitesimal generator of rotations though it belongs to $\mathcal{L}((E),(E)^*)$. In case of finite degree of freedom, the corresponding operator is $x_j\partial_k-x_k\partial_j$. Then by a simple calculation we obtain

Proposition 5.4

$$U(x_j\partial_k-x_k\partial_j)U^{-1}=x_j\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_j}.$$

Thus, as for infinitesimal generators of rotations, the exact form coincides with the formal analogy. But this is merely by good fortune.

Finally we consider the Fourier transform on white noise functionals introduced by Kuo [10], [12]. In fact, it is imbedded in a one-parameter group of Fourier-Mehler transforms which we shall discuss. For $\Phi\in(E)^*$ the Fourier-Mehler transform $\mathfrak{F}_\theta\Phi$, $\theta\in\mathbb{R}$, is defined by

$$S\mathfrak{F}_\theta\Phi(\xi)=S\Phi(e^{i\theta}\xi)\exp\left(\frac{i}{2}e^{i\theta}\sin\theta\,\langle\xi,\xi\rangle\right),\qquad \xi\in E_{\mathbb{C}}. \tag{21}$$

This implicit definition works well due to the characterization theorem of generalized white noise functionals, see [12] for details. It is known that $\mathfrak{F}_\theta \in \mathcal{L}((E)^*, (E)^*)$. The operator $\mathfrak{F} = \mathfrak{F}_{-\pi/2}$ is called *Kuo's Fourier transform.*

In order to study $U\mathfrak{F}_\theta U^{-1}$ we recall the (usual) Fourier-Mehler transform $\mathcal{F}_\theta$, $\theta \in \mathbb{R}$. Following [2, Chap.7], for $\theta \not\equiv 0 \pmod{\pi}$ we define

$$\mathcal{F}_\theta f(x) = \left(-2\pi i e^{i\theta} \sin\theta\right)^{-D/2} \int_{\mathbb{R}^D} f(y) \exp\left(\frac{-i(|x|^2+|y|^2)\cos\theta + 2i\langle x, y\rangle}{2\sin\theta}\right) dy. \tag{22}$$

For $\theta \equiv 0 \pmod{\pi}$ we put

$$\mathcal{F}_\theta f(x) = \begin{cases} f(x), & \theta \equiv 0 \pmod{2\pi}, \\ f(-x), & \theta \equiv \pi \pmod{2\pi}. \end{cases}$$

These operators are defined, for example on $L^1(\mathbb{R}^D, dx)$. Moreover, $\{\mathcal{F}_\theta\}_{\theta\in\mathbb{R}}$ becomes a one-parameter group of automorphisms of $\mathcal{S}(\mathbb{R}^D)$. It is noted that

$$\mathcal{F} = \mathcal{F}_{\pi/2}, \qquad \mathcal{F}^* = \mathcal{F}^{-1} = \mathcal{F}_{-\pi/2},$$

where $\mathcal{F}$ is the (usual) Fourier transform:

$$\mathcal{F}f(x) = \hat{f}(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} f(y) e^{i\langle x, y\rangle} dy.$$

Theorem 5.5 *It holds that*

$$U\mathfrak{F}_\theta U^{-1} = e^{-|x|^2/4} \circ \mathcal{F}_\theta \circ e^{|x|^2/4}.$$

In particular,

$$U\mathfrak{F}U^{-1} = e^{-|x|^2/4} \circ \mathcal{F}^* \circ e^{|x|^2/4}.$$

PROOF. Let $\Phi \in (E)^*$. Note first that

$$\begin{aligned} S\Phi(\xi) &= e^{-\langle\xi,\xi\rangle/2} \int_{\mathbb{R}^D} \Phi(x) e^{\langle x,\xi\rangle} \mu(dx) \\ &= e^{-\langle\xi,\xi\rangle/2} \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} \Phi(x) e^{-|x|^2/2} e^{i\langle x, -i\xi\rangle} dx, \end{aligned}$$

where the integrals are understood in the distribution sense, i.e., symbolic notation for bilinear forms. (This remark remains valid throughout the proof.) Then we have

$$S\Phi(\xi) = e^{-\langle\xi,\xi\rangle/2} (\Phi e^{-|x|^2/2})\hat{}\,(-i\xi), \qquad \xi \in \mathbb{C}^D, \tag{23}$$

where the Fourier transform is in the distribution sense. In view of (23) we have

$$\begin{aligned} S\mathfrak{F}_\theta\Phi(\xi) &= e^{-\langle\xi,\xi\rangle/2} (\mathfrak{F}_\theta\Phi \cdot e^{-|x|^2/2})\hat{}\,(-i\xi), \\ S\Phi(e^{i\theta}\xi) &= e^{-e^{2i\theta}\langle\xi,\xi\rangle/2} (\Phi \cdot e^{-|x|^2/2})\hat{}\,(-ie^{i\theta}\xi). \end{aligned}$$

Then (21) becomes

$$e^{-\langle\xi,\xi\rangle/2}(\mathfrak{F}_\theta\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-i\xi) =$$

$$= \exp\left(\frac{i}{2}e^{i\theta}\sin\theta\,\langle\xi,\,\xi\rangle\right) e^{-e^{2i\theta}\langle\xi,\xi\rangle/2}(\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-ie^{i\theta}\xi),$$

and therefore

$$\begin{aligned}
&(\mathfrak{F}_\theta\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-i\xi) = \\
&= \exp\left\{\left(\frac{i}{2}e^{i\theta}\sin\theta - \frac{e^{2i\theta}}{2} + \frac{1}{2}\right)\langle\xi,\,\xi\rangle\right\}(\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-ie^{i\theta}\xi) \\
&= \exp\left\{-\left(\frac{i}{2}e^{i\theta}\sin\theta\right)\langle\xi,\,\xi\rangle\right\}(\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-ie^{i\theta}\xi).
\end{aligned} \tag{24}$$

For simplity we put

$$\alpha = \alpha(\theta) = \frac{i}{2}e^{i\theta}\sin\theta = -\frac{1}{4} + \frac{1}{4}e^{2i\theta}.$$

Note that

$$\mathrm{Re}\,\alpha \le 0 \qquad \text{and} \qquad \mathrm{Re}\,\alpha = 0 \Longleftrightarrow \alpha = 0 \Longleftrightarrow \theta \equiv 0 \pmod{\pi}.$$

Then, (24) becomes

$$(\mathfrak{F}_\theta\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-i\xi) = e^{-\alpha\langle\xi,\xi\rangle}(\Phi\cdot e^{-|x|^2/2})\widehat{\ }(-ie^{i\theta}\xi),$$

and hence

$$(\mathfrak{F}_\theta\Phi\cdot e^{-|x|^2/2})\widehat{\ }(\xi) = e^{\alpha\langle\xi,\xi\rangle}(\Phi\cdot e^{-|x|^2/2})\widehat{\ }(e^{i\theta}\xi), \qquad \xi\in\mathbb{C}^D. \tag{25}$$

Applying the inverse Fourier transform to (25), we obtain

$$\begin{aligned}
&\mathfrak{F}_\theta\Phi(x)e^{-|x|^2/2} = \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} e^{\alpha\langle\xi,\xi\rangle}(\Phi\cdot e^{-|y|^2/2})\widehat{\ }(e^{i\theta}\xi)e^{-i\langle x,\xi\rangle}d\xi \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D}\left\{\left(\frac{1}{\sqrt{2\pi}}\right)^D\int_{\mathbb{R}^D}\Phi(y)e^{-|y|^2/2}e^{i\left\langle y,\,e^{i\theta}\xi\right\rangle}dy\right\}e^{\alpha\langle\xi,\xi\rangle-i\langle x,\xi\rangle}d\xi \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D}\Phi(y)e^{-|y|^2/2} \\
&\qquad\times\left\{\left(\frac{1}{\sqrt{2\pi}}\right)^D\int_{\mathbb{R}^D}\exp\left(\alpha\langle\xi,\,\xi\rangle - i\langle x,\,\xi\rangle + i\left\langle y,\,e^{i\theta}\xi\right\rangle\right)d\xi\right\}dy.
\end{aligned}$$

As is easily seen,

$$\left(\frac{1}{\sqrt{2\pi}}\right)^D\int_{\mathbb{R}^D}\exp\left(\alpha\langle\xi,\,\xi\rangle + i\left\langle y,\,e^{i\theta}\xi\right\rangle - i\langle x,\,\xi\rangle\right)d\xi =$$

$$= \begin{cases} (-2\alpha)^{-D/2} \exp\left(\dfrac{\left\langle x - e^{i\theta}y,\, x - e^{i\theta}y \right\rangle}{4\alpha}\right), & \theta \not\equiv 0 \pmod{\pi}, \\ (2\pi)^{D/2}\delta_y(x), & \theta \equiv 0 \pmod{2\pi}, \\ (2\pi)^{D/2}\delta_{-y}(x), & \theta \equiv \pi \pmod{2\pi}. \end{cases}$$

Suppose first that $\theta \not\equiv 0 \pmod{\pi}$. Then we obtain

$$\mathfrak{F}_\theta \Phi(x) e^{-|x|^2/2} =$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^D (-2\alpha)^{-D/2} \int_{\mathbb{R}^D} \Phi(y) e^{-|y|^2/2} \exp\left(\frac{\left\langle x - e^{i\theta}y,\, x - e^{i\theta}y \right\rangle}{4\alpha}\right) dy. \tag{26}$$

Since

$$\begin{aligned} \frac{\left\langle x - e^{i\theta}y,\, x - e^{i\theta}y \right\rangle}{4\alpha} &= \frac{|x|^2 - 2e^{i\theta}\langle x, y\rangle + e^{2i\theta}|y|^2}{2ie^{i\theta}\sin\theta} \\ &= \frac{e^{-i\theta}|x|^2 + e^{i\theta}|y|^2 - 2\langle x, y\rangle}{2i\sin\theta} \\ &= \frac{-i(|x|^2 + |y|^2)\cos\theta + 2i\langle x, y\rangle}{2\sin\theta} - \frac{|x|^2 - |y|^2}{2}, \end{aligned}$$

(26) becomes

$$\mathfrak{F}_\theta \Phi(x) e^{-|x|^2/2} =$$

$$\begin{aligned} &= e^{-|x|^2/2} \left(\frac{1}{\sqrt{2\pi}}\right)^D (-2\alpha)^{-D/2} \\ &\quad \times \int_{\mathbb{R}^D} \Phi(y) \exp\left(\frac{-i(|x|^2 + |y|^2)\cos\theta + 2i\langle x, y\rangle}{2\sin\theta}\right) dy \\ &= e^{-|x|^2/2} \left(-2\pi i e^{i\theta}\sin\theta\right)^{-D/2} \int_{\mathbb{R}^D} \Phi(y) \exp\left(\frac{-i(|x|^2 + |y|^2)\cos\theta + 2i\langle x, y\rangle}{2\sin\theta}\right) dy. \end{aligned}$$

In view of the definition (22) we have

$$\mathfrak{F}_\theta \Phi(x) e^{-|x|^2/2} = e^{-|x|^2/2} \mathcal{F}_\theta \Phi(x),$$

namely,

$$\mathfrak{F}_\theta = \mathcal{F}_\theta. \tag{27}$$

As is easily verified, (27) is valid also for $\theta \equiv 0 \pmod{\pi}$. Consequently, for any $\theta \in \mathbb{R}$

$$U \mathfrak{F}_\theta U^{-1} = e^{-|x|^2/4} \circ \mathcal{F}_\theta \circ e^{|x|^2/4},$$

as desired. qed

In fact, Kuo found the white noise version of Fourier-Mehler transform in the above way though our discussion is reversed. The key idea is the identity (27).

It is known that $\mathfrak{F} = \mathfrak{F}_{-\pi/2}$ is characterized as a unique continuous operator from $(E)^*$ into itself such that

$$\mathfrak{F}\partial_t = ix(t)\mathfrak{F}, \qquad \mathfrak{F}x(t) = i\partial_t\mathfrak{F}.$$

(More precisely, the operators ∂_t and $x(t)$ should be replaced with smeared ones because they are not operators on $(E)^*$. For details see [3] where the intertwining properties of the Fourier-Mehler transform is discussed as well.) Therefore, the operator

$$\tilde{\mathfrak{F}} = U\mathfrak{F}U^{-1}$$

is charactreized by the following intertwining properties:

$$\tilde{\mathfrak{F}}\left(\frac{x_j}{2} + \frac{\partial}{\partial x_j}\right) = ix_j\tilde{\mathfrak{F}}, \qquad \tilde{\mathfrak{F}}x_j = i\left(\frac{x_j}{2} + \frac{\partial}{\partial x_j}\right)\tilde{\mathfrak{F}}. \tag{28}$$

On the other hand, the usual Fourier transform $\mathcal{F}^*$ on $\mathcal{S}'(\mathbb{R}^D)$ is defined by

$$(\mathcal{F}^*f)(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^D \int_{\mathbb{R}^D} f(y)e^{-i\langle x,y\rangle}dy$$

in the distribution sense and satisfies

$$\mathcal{F}^*\frac{\partial}{\partial x_j} = ix_j\mathcal{F}^*, \qquad \mathcal{F}^*x_j = i\frac{\partial}{\partial x_j}\mathcal{F}^*.$$

This is compared with (28).

6 Appendix

We summarize the above discussion into the following "translation table." In the left column we list general notation of white noise calculus and in the middle the corresponding expressions derived from white noise calculus with finite degree of freedom via the unitary map (12). In the right column we list formally expected notions of usual finite dimensional calculus.

TRANSLATION TABLE

white noise calculus in general	finite degree of freedom (exact translation)	conventional formal analogy
(T,ν)	$T = \{1,2,\cdots,D\}$ with counting measure	
(E^*,μ)	$(\mathbb{R}^D, dx)$	
$(E) \subset L^2(E^*,\mu) \subset (E)^*$	$\mathcal{D} \subset L^2(\mathbb{R}^D,dx) \subset \mathcal{D}^*$	$\mathcal{S}(\mathbb{R}^D) \subset L^2(\mathbb{R}^D,dx) \subset \mathcal{S}'(\mathbb{R}^D)$

∂_t	$\dfrac{x_j}{2} + \dfrac{\partial}{\partial x_j}$	$\dfrac{\partial}{\partial x_j}$
∂_t^*	$\dfrac{x_j}{2} - \dfrac{\partial}{\partial x_j}$	$\left(\dfrac{\partial}{\partial x_j}\right)^* = -\dfrac{\partial}{\partial x_j}$
$E^* \ni x \mapsto x(t)$	$\mathbb{R}^D \ni x \mapsto x_j$	
$x(t) = \partial_t + \partial_t^*$	x_j (as multiplication operator)	
$N = \int_T \partial_t^* \partial_t dt$	$\sum_{j=1}^D \left(-\dfrac{\partial^2}{\partial x_j^2} + \dfrac{x_j^2}{4} - \dfrac{1}{2}\right)$	$\sum_{j=1}^D \left(\dfrac{\partial}{\partial x_j}\right)^* \dfrac{\partial}{\partial x_j}$
$\Delta_G = \int_T \partial_t^2 dt$	$\sum_{j=1}^D \left(\dfrac{\partial^2}{\partial x_j^2} + x_j \dfrac{\partial}{\partial x_j} + \dfrac{x_j^2}{4} + \dfrac{1}{2}\right)$	$\sum_{j=1}^D \dfrac{\partial^2}{\partial x_j^2}$
$\langle :x^{\otimes 2}:, \tau\rangle = \int_T :x(t)^2: dt$	$\sum_{j=1}^D (x_j^2 - 1)$	$\sum_{j=1}^D x_j^2$
$x(s)\partial_t - x(t)\partial_s$	$x_j \dfrac{\partial}{\partial x_k} - x_k \dfrac{\partial}{\partial x_j}$	
$\mathfrak{F}_\theta : (E)^* \to (E)^*$	$e^{-\lvert x\rvert^2/4} \circ \mathcal{F}_\theta \circ e^{\lvert x\rvert^2/4}$	$\mathcal{F}_\theta^* : \mathcal{S}' \to \mathcal{S}'$
$\mathfrak{F} : (E)^* \to (E)^*$	$e^{-\lvert x\rvert^2/4} \circ \mathcal{F}^* \circ e^{\lvert x\rvert^2/4}$	$\mathcal{F}^* : \mathcal{S}' \to \mathcal{S}'$

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